A cork model for parking

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The problem

- Many facilities, such as department stores, outlet villages, big theaters, airports, sport resorts and so on, have a dedicated parking area, usually with a slightly oversized capacity for their users/customers.
- Parking in this compounds may be time consuming, according to the arrival time.
- The more the parking area gets congested, the harder to find an empty lot and to park in it.
- As a result users may weight up arriving late or early to destination to avoid congestion in the parking area.
- We call this situation *cork model* in analogy of the bottleneck model

Literature review

1. Vickrey (1963)
2. Arnott, de Palma & Lindsey (1990, 1993, 1999)
3. Arnott & Rowse (1999)
4. Voith (1998)
5. Ommeren, Wentink & Dekkers (2011)
6. Anderson & de Palma (2004)
7. Danielis & Marcucci (2002)

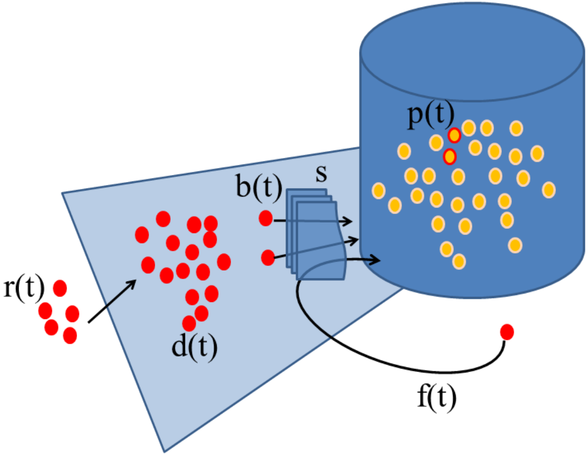
Main results

- Similar but not identical properties of the ‘standard’ bottleneck model
- Congestion at the gate can be eliminated but not congestion in the Parking Area
- The average arrival time at the Parking Area is before the preferred time
- The optimal average arrival time at the Parking Area is before the average arrival time

Assumptions

- M identical users are assumed to be equally interested to get at the Destination at the ideal time 0.
- When reaching the Parking Area at time t, the user may eventually spend time ($q \geq 0$) to queue before entering the Parking Area and time ($c \geq 0$) to cruise for parking in the Parking Area.
- We normalize to zero the time spent to reach the Parking Area as well as the walking time to reach the final destination from the parking lot.
- The user pay a price per minute α for the time $h = q + c$ spent to park.
- The arrival time at Destination is: $l = t + q + c$, i.e. the lag ($l > 0$) or lead ($l < 0$). Arriving early or late has a cost per minute β.
- We assume that: $\alpha > \beta$.

Parking procedure



Parking procedure

- At each instant t,
- $r(t)$ cars approach to a Parking Area
- $d(t)$ the total number of cars that aim at parking
- $f(t)$ cars that are already in the Parking Area and failed to find an empty parking lot.
- $b(t)$ new cars in the Parking Area
- s be the maximum number of cars that can simultaneously search for a parking lot in the Parking Area
- $\rho(t)$ is percentage of free parking lots in the Parking Area that will be available for the user entering at t.
- $\rho(t)\, g(t)$ cars that succeed in finding an empty parking lot
- $f(t) = (1 - \rho(t))\, g(t)$ cars that fail.

Drivers

Drivers can be in one of the four cases: not yet arrived, waiting in front of the gate, cycling in the Parking Area, Parked:
$q(t)$ queuing time is:

$$q(t) = \frac{\max\{d(t), b(t)\} - b(t)}{b(t)}$$

$c(t)$ expected cycling time. The first driver finds an empty parking lot for sure and therefore $c(t_I) = 1$ (one attempt). Since the Parking Area gets crowded and crowded over time, c increases over time. The maximum value is:

$$\bar{c} = \frac{N}{N - M}.$$

Therefore, the lag or lead time is:

$$l(t) = t + [q(t) + c(t + q(t))] = t + \left[q(t) + \frac{1}{\rho(t + q(t))}\right]$$

Cars parked in the Parking area I

Let $P(t)$ be the total number of parked car and $g(t) = f(t) + b(t)$ the number of cars searching for an empty parking lot in the Parking Area. The first order derivative of $P(t)$ is given by:

$$P'(t) := p(t) = \rho(t)\, g(t) = \frac{N - P(t)}{N} \times g(t) \tag{1}$$

Since all users share the same optimal arrival time and are rational, there exists a time window $[t_I, t_F]$, such that $g(t) = s$ for all $t \in [t_I, t_F]$. In the rush hour the number of searching for an empty parking lot is given by the absortive capacity of the Parking.

Solving the differential equation (1):

$$P(t) = N + ke^{-\frac{1}{N}st}$$

To compute the value for the constant k, we shall assume that at the time of the first arrival, the parking area is empty, that is $\exists t_I$ such that

$$P(t_I + 1) = N + ke^{-\frac{1}{N}st_I} = 0$$

Solving for k we get

$$k = -\frac{N}{e^{-\frac{1}{N}st_I+1}}$$

Therefore we can write:

$$P(t) = N\left(1 - e^{\frac{s}{N}(t_I - t + 1)}\right) \tag{2}$$

Cars parked in the Parking area III

The rush hour will last until the last user arriving at time t_F parks. Since the last user will face no queue at the gate, $q(t_F) = 0$, and maximum time cruising for parking, $c(t_F) = \bar{c} = \frac{N}{N-M}$, we must have that

$$P(t_F + c(t_F)) = N\left(1 - e^{\frac{s}{N}\left(t_I - t_F - \frac{M}{N-M}\right)}\right) = M$$

and we can solve the last equation to get

$$t_F = t_I + (1 - \bar{c}) + \frac{N}{s} \ln \bar{c} \tag{3}$$

From equation (3) it can be seen that the length of the rush hour depends only on the total number of users and the technical characteristics of the Parking Area, measured by the total number of parking lots, N, and the absorptive capacity of the Parking Area, s.

Cars parked in the Parking area IV

The total cost for a driver arriving at the Parking Area at time $t \in [t_I, t_F]$ is

$$TC(t) = \alpha\left(q(t) + c(t + q(t))\right) + \beta\left|l(t + q(t))\right| \qquad (4)$$

Note: at time t_I: $q(t_I) = 0$; $c(t_I) = 1$ and $l(t_I) = t_I + 1 < 0$ and at time t_F: $q(t_F) = 0$, but $c(t_F) = \bar{c}$ and $l(t_F) = t_F + \bar{c}$. Now we can compute the value of t_I assuming that, at equilibrium, the first and the last user must be equally well off

$$\alpha + \beta\left|t_I + 1\right| = \alpha\bar{c} + \beta\left|\left[t_I + (1 - \bar{c}) + \frac{N}{s}\ln\bar{c}\right] + \bar{c}\right| \qquad (5)$$

Therefore, solving equation (5) we can compute

$$t_I = -\frac{1}{2}\left(\frac{N}{s}\ln\bar{c} + \frac{\alpha}{\beta}(\bar{c} - 1)\right) - 1 < 0 \qquad (6)$$

$$t_F = \frac{1}{2}\left(\frac{N}{s}\ln\bar{c} - \frac{\alpha}{\beta}(\bar{c} - 1)\right) - \bar{c} \qquad (7)$$

Average arrival time and preferred time

Remark

From equations (6) and (7) it comes immediately that the rush hour is not centered at 0*, the optimal desired time of arrival. Classical bottleneck models usually provide a symmetric equilibrium interval. The mean value of the interval* $[t_I, t_F]$ *is:*

$$t_{av} = -\frac{1}{2}\frac{\alpha}{\beta}\left(\bar{c}-1\right) - \frac{1}{2}\left(\bar{c}+1\right) < 0$$

Remark

It is also interesting to note that the mean between the maximum lead, $l\left(t_I\right)$*, and the maximum lag,* $l\left(t_F\right)$*, is still negative:*

$$l_{av} = -\frac{\alpha}{\beta}\left(\bar{c}-1\right) < 0.$$

Total number of cars entered in the Parking Area at time t

Total number of cars $B(t)$ includes both cars that have already parked and those still cruising for parking. Since: $b(t) = s - f(t) = \rho(t)\, s$, therefore:

$$B(t) = \int_{t_l}^{t} b(x)\, dx = s \int_{t_l}^{t} \rho(x)\, dx = s \int_{t_l}^{t} \frac{N - P(x)}{N} dx = P(t) \quad (8)$$

$$= s \int_{t_l}^{t} e^{\frac{s}{N}(t_l - x + 1)} dx = N e^{\frac{s}{N}} \left(1 - e^{\frac{s}{N}(t_l - t)}\right) \quad (9)$$

Cars cycling in the Parking Area at time t

Noting that $p(t) = P'(t) \simeq \frac{B(t)-P(t)}{c(t)}$, we obtain:

$$\begin{aligned} c(t) &\simeq \frac{B(t) - P(t)}{P'(t)} = \frac{Ne^{\frac{s}{N}}\left(1 - e^{\frac{s}{N}(t_l - t)}\right) - N\left(1 - e^{\frac{s}{N}(t_l - t + 1)}\right)}{se^{\frac{s}{N}(t_l - t + 1)}} \\ &\simeq \frac{N}{s} e^{-\frac{s}{N}(t_l - t)}\left(1 - e^{-\frac{s}{N}}\right) \end{aligned}$$

Cars parked in the Parking Area at time t I

In order to compute the cars parked at the Parking Area at time t, we consider the lag or lead time that makes indifferent drivers.
We recall that:

$$l(t) = t + q(t) + c(t + q(t)) \begin{array}{ll} > 0 & \text{lag} \\ < 0 & \text{lead} \end{array}$$

Let $h(t) = q(t) + c(t + q(t))$, hence, all drivers arriving at time t must be equally well off, that is:

$$\alpha - \beta(t_l + 1) = \alpha h(t) + \beta |t + h(t)| . \tag{10}$$

Cars parked in the Parking Area at time t II

Let $\tilde{t}$ the arrival time at the gate of the driver exactly in time. Then for those drivers arriving at t such that $t_I \leq t < \tilde{t}$ or $t + h(t) < 0$, it follows:

$$\alpha - \beta (t_I + 1) = \alpha h(t) - \beta (t + h(t))$$

or:

$$h(t) = 1 + \frac{\beta}{\alpha - \beta} (t - t_I)$$

For those drivers arriving at time $\tilde{t} < t \leq t_F$ or $t + h(t) > 0$, it follows:

$$\alpha - \beta (t_I + 1) = \alpha h(t) - \beta (t + h(t))$$

or:

$$h(t) = 1 - \frac{\beta}{\alpha + \beta} (t + t_I + 2)$$

Driver exactly in time

Finally, we can easily find the arrival time $\tilde{t}$ that allows drivers to be exactly in time, $\tilde{t} + h(\tilde{t}) = 0$ at the destination:

$$\alpha - \beta (t_l + 1) = \alpha h (\tilde{t})$$

or:

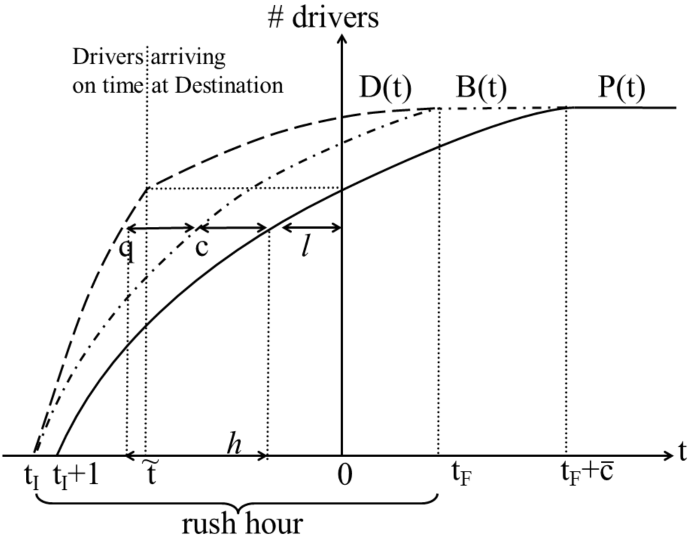
$$\tilde{t} = \frac{\beta}{\alpha} (t_l + 1) - 1 < 0$$

Total cars at the gates of the Parking Area at time t

Since $D(t) = P(t + h(t))$, we have:

$$D(t) = \begin{cases} N\left(1 - e^{-\frac{s}{N}\frac{\alpha(t-t_I)}{\alpha-\beta}}\right) & t_I \leq t \leq \tilde{t} \\ N\left(1 - e^{-\frac{s}{N}\frac{\alpha(t-t_I)-2\beta(t_I+1)}{\alpha+\beta}}\right) & \tilde{t} < t \leq t_F \end{cases}$$

Graphical representation



Characteristics of the market eq. with no parking fees

Market equilibrium is inefficient for two reasons:

1. queueing at the gate is a pure loss of time;
2. because cruising costs are increasing over time, drivers would arrive too early.

Characteristics of the market eq. with parking fees

In order to find the optimal parking schedule, we need to minimizes the total cost of queueing, parking and the lag/lead time. The cumulate cruising time in the Parking Area is given by technical characteristics and it cannot be modified.

Since queueing time is a pure deadweight loss, social optimum requires to reduce it to zero. Hence, drivers should be scheduled to arrive at such rate that $d(t) = b(t)$, for all $t_I \leq t \leq t_F$ and choosing the optimal time of the first arrival, such that:

$$\min_{t_I} \int_{t_I+1}^{t_I+1+\frac{N}{s}\ln \bar{c}} |x| \, dP(x)$$

After integrating, we obtain:

$$t_I^{SO} = \frac{N}{s} \ln \frac{1+c}{2c} - 1 \tag{11}$$

Optimal fees

Using equation (4), and previous equations, the total costs (excluding parking fees) is:

$$TC(t) = \begin{cases} (\alpha - \beta)\, c(t) - \beta t & t_I^{SO} \le t \le \tilde{t}^{SO} \\ (\alpha + \beta)\, c(t) + \beta t & \tilde{t}^{SO} < t \le t_F^{SO} \end{cases}$$

The optimal fee requires that all consumers should be well off. The first driver faces a total cost (excluding the parking fee) equals to $TC\left(t_I^{SO}\right) = \alpha + \beta \frac{1}{1+\bar{c}} \frac{N}{s} \ln \bar{c}$ and the last driver $TC\left(t_F^{SO}\right) = \left(\alpha + \beta \frac{1}{1+\bar{c}} \frac{N}{s} \ln \bar{c}\right) \bar{c} = T\bar{C}$.

Theorem

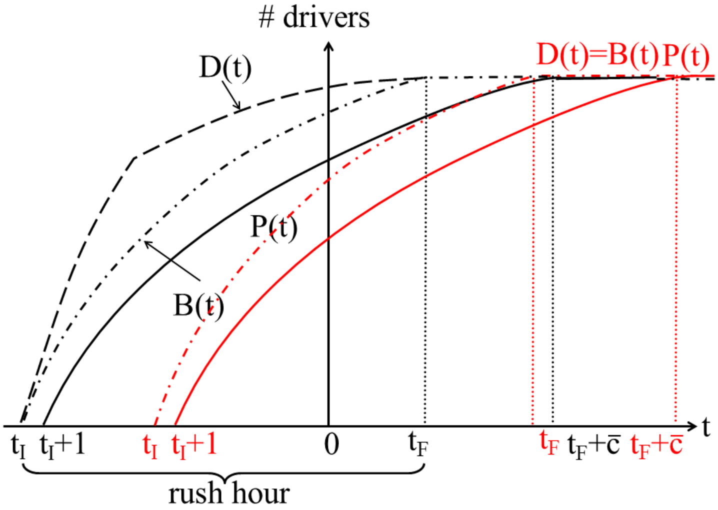
For any driver, the optimal fee is of the form: $\varphi(t) = T\bar{C} - TC(t)$. *That is:*

$$\begin{aligned} \varphi(0) &= \frac{M}{N-M}\left(\alpha + \beta \frac{1}{1+\bar{c}} \frac{N}{s} \ln \bar{c}\right) \\ \varphi'(t) &= -TC'(t) \end{aligned}$$

Remark

$t_I^{SO} > t_I^*$ *and* $t_F^{SO} > t_F^*$

Graphical presentation



Conclusions

- Similar but not identical properties of the 'standard' bottleneck model
- Congestion at the gate can be eliminated but not congestion in the Parking Area
- The average arrival time at the Parking Area is before the preferred time
- Market equilibrium is inefficient for two reasons:

1. queueing at the gate is a pure loss of time;
2. because cruising costs are increasing over time, drivers would arrive too early.

- The optimal average arrival time at the Parking Area is before the average arrival time